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MATHEMATICAL MODELS OF PHYSICAL PHENOMENA

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CHAPTER I

INTRODUCTION

Even a simple physical phenomenon involves so many details and so many unknown features that complete analysis is usually out of the question. Hence, a quantitative investigation normally begins with the construction of a mathematical idealization which is intended to account for those characteristics of the physical system which are decisive in describing its behavior to the degree of completeness necessary for the purpose in mind. This idealization is commonly called a mathematical model.

The derivations of such models which appear in technological books frequently contain logical gaps which are confusing to a beginner. Assumptions (either mathematical or physical) which guide the derivation may not be clearly or fully stated. Reasoning couched in mathematical terms is often so vague that its intended meaning is undecipherable; and if the apparent meaning is the intended meaning, then the reasoning is faulty (a prime example of this vagueness is the indiscriminate use of differentials and an accompanying failure to indicate the arguments on which functions depend). In fact, an author's success in arriving at the generally accepted answer often appears to be less dependent on the validity of the logic than on knowing the answer in advance.

In analyzing physical phenomena it is possible to avoid these pitfalls, and the purpose of this study is to show how the use of a few simple and well-known theorems and some care in the application of mathematical concepts can clarify the muddle that so often precedes the model. Of course, "clarify" is used here in a special sense -- namely, to cope with the logical difficulties encountered and to resolve them by a rational process. The object of such efforts is to give derivations which are satisfying and reliable, because they furnish models that are logical consequences of the assumptions and approximations on which they are based.

It is obviously impossible to establish a procedure, entirely applicable under all circumstances, for the derivation of a mathematical model. Every effective derivation, however, contains certain salient features which will now be noted and explained. The attention of the reader is invited to them as they occur in the various derivations of mathematical models presented in this study.

(1) Physical Assumptions. The principles which are usually thought of as physical laws are actually approximations of the real world stated in mathematical terms. If a mathematical model of a physical system is to be derived, then all the physical assumptions made about the system must be formulated so that the appropriate physical laws may be selected and employed in the derivation.

(2) Mathematical Assumptions and Nomenclature. The mathematical symbolism used in a derivation should be presented. In addition, all mathematical assumptions should be clearly and unambiguously stated so that one may know whether the hypotheses of the theorems necessary to

justify the derivation are unequivocally satisfied. For example, a certain function may be required to be continuously differentiable.

(3) Formal Derivation. The formal mathematical steps themselves should be logical consequences of clearly stated physical and mathematical assumptions. The crucial devices used in the derivation are the so-called model-building steps, which normally hinge on physical rather than mathematical reasoning. They should be made as simple and unchallengeable as possible, so that a reader will not get the impression that new assumptions have been surreptitiously introduced.

(4) The Model. The derivation terminates with the mathematical model, which is supposedly an accurate representation of the physical model depicted by the assumptions given at the outset. The efficacy of the model depends, of course, on the validity of the original physical assumptions.

Several examples are presented in the succeeding chapters of this thesis to illustrate the ideas just discussed and to demonstrate the use of well-known theorems and mathematical techniques in the derivations of mathematical models of physical phenomena. Chapters II and III are concerned with mechanical systems involving linear springs whose distributed masses are not neglected. The example of Chapter II is a static system, and the corresponding model is relatively simple. The derivation of the model involves the consideration of a function of one real variable. Chapter II contains the only example in the thesis in which the mathematical model is analyzed. The example of Chapter III, in contrast, is a dynamic system, and the relevant mathematical model is considerably more complicated. The construction of the model requires techniques more subtle than those in Chapter II; and the complete derivation is not given, since a large portion of it is simply the application of routine methods of the calculus of variations. Only that part of the derivation which serves the purpose of this study is presented. Chapter IV is a "patching up" of a derivation given in a physics text for the dynamical equations of a perfect fluid. The object of this chapter is to illustrate how a higher-dimensional law of the mean is used in that process. Derivations of the wave equation, the heat equation, and Laplace's equation -- three classical differential equations -- are given in Chapter V. These equations are found as the mathematical models corresponding to three easily visualized physical systems.

The object of Chapter VI is to illustrate an error typical of those which occur in various derivations of the mathematical models of physical phenomena and to suggest two ways of rectifying such errors. Oddly enough, many of the derivations which contain errors of the types discussed in this study are accompanied by correct results; and perhaps no further comment is required beyond the observation that the providence which explains this anomaly deserves man's gratitude.

CHAPTER II

STATIC SYSTEM WITH DISTRIBUTED MASS

A uniform linear spring having total weight W lb and spring constant K lb/in is L in long when lying on its side. The spring is placed on end, and its length after it has been up-ended is to be found. It is assumed that the up-ended spring has reached a state of motionless equilibrium so that a nonuniform but time-independent compression exists throughout its length.

Let x denote the distance measured along the spring from its left end toward the right when the spring is lying on its side (Fig. 1). For each $x \in [0,L]$ let $y(x)$ be the length of the up-ended segment which originally occupied the interval $[0,x]$, where y is measured vertically downward from the top of the up-ended spring (Fig. 2(a)). Assume throughout the construction and analysis of the mathematical model that y is a differentiable function of x on $[0,L]$, and stipulate that $y(0) = 0$ (this stipulation is actually a model-building step, since it implies that the left end of the horizontal spring becomes the top of the spring when it is placed on end).

Let $x_0 \in [0,L]$ be a fixed, arbitrary distance measured along the spring lying on its side. Consider the section $[0,x_0]$ of the spring. The spring constant for this section is $\frac{KL}{x_0}$. Let $(x_0 + h) \in [0,L]$, where $h \neq 0$. Two cases present themselves.

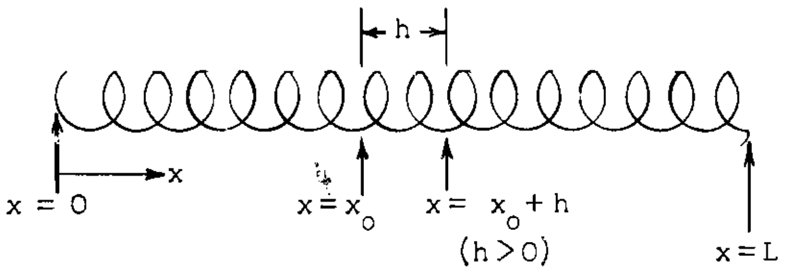

Figure 1. Spring in the Initial State

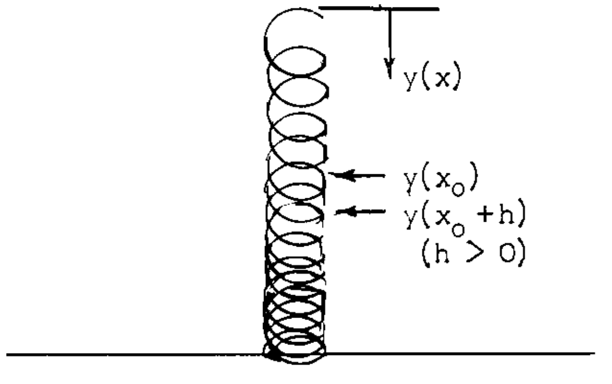

Figure 2(a). Up-Ended Spring

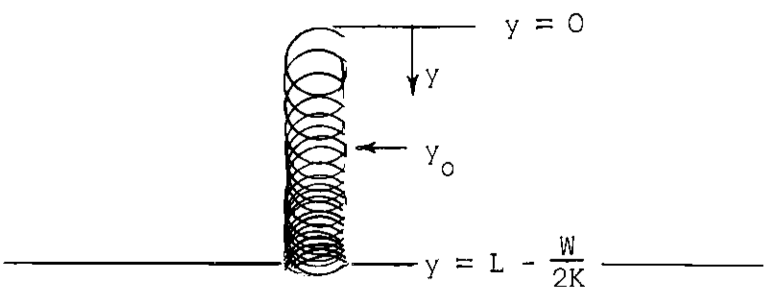

Figure 2(b). Up-Ended Spring

Case I: suppose $h > 0$. When the spring is up-ended, the segment of the spring which initially occupied the interval $[x_o, x_o + h]$ is shortened, or compressed. The amount of this compression is denoted by $u_h(x_o)$ which is given by the equation

$$u_h(x_o) = h - [y(x_o + h) - y(x_o)]. \qquad (1)$$

The question now arises as to the force which causes the compression $u_h(x_o)$. It is here that the critical assumption involved in the derivation of the mathematical model is made. It is assumed that $u_h(x_o)$ lies between the following bounds:

(a) that compression, denoted by $u_h^{(a)}$, which would result if the force exerted on the segment $[x_o, x_o + h]$ were due only to the weight of the section $[0,x_o]$, where the weight of $[x_o, x_o + h]$ is neglected;

(b) that compression, denoted by $u_h^{(b)}$, which would result if the weight of the entire section $[0, x_o + h]$ acted on a massless segment $[x_o, x_o + h]$.

That is, the following inequality is asserted:

$$u_h^{(a)} \leq u_h(x_o) \leq u_h^{(b)}. \qquad (2)$$

The quantities $u_h^{(a)}$ and $u_h^{(b)}$ are computed in the succeeding steps of this paragraph. Consider the alternative (a). Then the compressive force acting on the segment $[y(x_o), y(x_o + h)]$ is

$$F_h^{(a)} = \frac{W\, x_o}{L} . \qquad (3a)$$

The resulting compression $u_h^{(a)}$ is the quotient of $F_h^{(a)}$ by the spring constant of the segment:

$$u_h^{(a)} = \frac{F_h^{(a)}}{KL/h}$$

$$= \frac{W\,x_o}{L}\left(\frac{h}{K\,L}\right)$$

$$= \frac{W\,x_o\,h}{K\,L^2}\,. \tag{4}$$

On the other hand, for alternative (b), the compressive force is

$$F_h^{(b)} = \frac{(x_o + h)\;W}{L}\,, \tag{3b}$$

and the resulting compression is

$$u_h^{(b)} = \frac{F_h^{(b)}}{KL/h}$$

$$= \frac{(x_o + h)\;W\,h}{K\,L^2}\,. \tag{5}$$

Equations (1), (4), and (5), and inequality (2) yield the following inequality

$$\frac{h\,W\,x_o}{K\,L^2} \leq h - [y(x_o + h) - y(x_o)] \leq \frac{h\,W\,(x_o + h)}{KL^2}\,,$$

or

$$\frac{W\,x_o}{K\,L^2} \leq 1 - \frac{y(x_o + h) - y(x_o)}{h} \leq \frac{W\,(x_o + h)}{K\,L^2}\,. \tag{6}$$

<u>Case II</u>: suppose $h < 0$. Here an analogous argument leads to the inequality

$$\frac{W(x_o + h)}{KL^2} \leq 1 - \frac{y(x_o + h) - y(x_o)}{h} \leq \frac{Wx_o}{KL^2} . \tag{7}$$

From inequalities (6) and (7) the following result is obtained:

$$\begin{aligned}\lim_{h \to 0^-} \left[1 - \frac{y(x_o + h) - y(x_o)}{h}\right] &= \lim_{h \to 0^+} \left[1 - \frac{y(x_o + h) - y(x_o)}{h}\right] \\ &= \lim_{h \to 0} \left[1 - \frac{y(x_o + h) - y(x_o)}{h}\right] \\ &= \frac{Wx_o}{KL^2} .\end{aligned}$$

Since y is a differentiable function of x on $[0,L]$, it follows that

$$1 - y'(x_o) = \frac{Wx_o}{KL^2} .$$

The choice of $x_o \varepsilon [0,L]$ was arbitrary. Thus, the following differential equation for all $x \varepsilon [0,L]$ arises:

$$y'(x) = 1 - \frac{Wx}{KL^2} , \tag{8}$$

which, together with the boundary condition $y(0) = 0$, constitutes the mathematical model desired. By using Equation (8) it is possible to deduce a restriction which must be placed on the relationship between W, K, and L if the model is to apply. Since physical considerations demand that y be a monotonically increasing function of x on $[0,L]$, it is necessary that $y'(x) \geq 0$ for all $x \varepsilon [0,L]$; i.e.,

$$1 - \frac{Wx}{KL^2} \geq 0, \qquad x \varepsilon [0,L];$$

or

$$\frac{Wx}{KL^2} \leq 1 .$$

In particular, this inequality must hold when $x = L$. Thus,

$$\frac{W}{KL} \leq 1.$$

This condition is necessary if Equation (8) is to be the proper mathematical model for the physical situation under discussion.

It is now necessary to analyze the mathematical model. The general solution of the differential equation (8) is

$$y(x) = x - \frac{Wx^2}{2KL^2} + C.$$

The value of the constant C may be determined by applying the required boundary condition $y(0) = 0$. Thus, it is found that $C = 0$, and it follows that

$$y(x) = x - \frac{Wx^2}{2KL^2}, \qquad 0 \leq x \leq L . \tag{9}$$

The length of the up-ended spring is $y(L)$, which is computed from (9):

$$y(L) = L - \frac{W}{2K} \text{ in.}$$

The problem is now solved. Note that the same result would have been obtained if the distributed mass of the spring had been approximated by a mass of weight $\frac{W}{2}$ concentrated at the upper end.

By extending the analysis slightly, additional interesting results can be obtained. Thus, if y has the same meaning as above, then the auxiliary problem of finding the linear density of the up-ended spring as a function of y may be considered.

Let $y_o \varepsilon [0, L - \frac{W}{2K}]$ denote the position of a point on the up-ended spring (Fig. 2b). Let $\Omega(y)$ denote the weight of the segment $[0,y]$ of this spring. Then the linear density ρ at y_o is defined as

$$\rho(y_o) = \lim_{h \to 0} [\frac{\Omega[y_o + h) - \Omega(y_o)}{h}] = \Omega'(y_o) ,$$

where it is assumed that Ω is a differentiable function of y on $[0, L - \frac{W}{2K}]$. But y_o was arbitrarily chosen. Hence,

$$\Omega'(y) = \rho(y), \qquad 0 \leq y \leq L - \frac{W}{2K} . \tag{10}$$

However, if x is written as a function of y, then

$$\Omega(y) = \frac{W\, x(y)}{L} ;$$

and from (10)

$$\rho(y) = \frac{W\, x'(y)}{L} , \qquad 0 \leq y \leq L - \frac{W}{2K} , \tag{11}$$

provided x can be expressed as a differentiable function of y.

The problem of finding a differentiable inverse $x(y)$ is now undertaken. In order for the function y, defined by Equation (9), to possess an inverse, it must be assured that y is a one-to-one function on its domain $[0,L]$. Consider a function $\bar{y}$ defined by the relation

$$\bar{y}(x) = x - \frac{Wx^2}{2KL^2}, \quad -\infty < x < +\infty .$$

Thus, $\bar{y}(x)$ coincides with $y(x)$ on $[0,L]$, but it has a larger domain of definition. The maximum of $\bar{y}$ occurs at $x = \frac{KL^2}{W}$ (Fig.3), and $\bar{y}$ is not one-to-one because its domain includes values of x greater than $\frac{KL^2}{W}$.

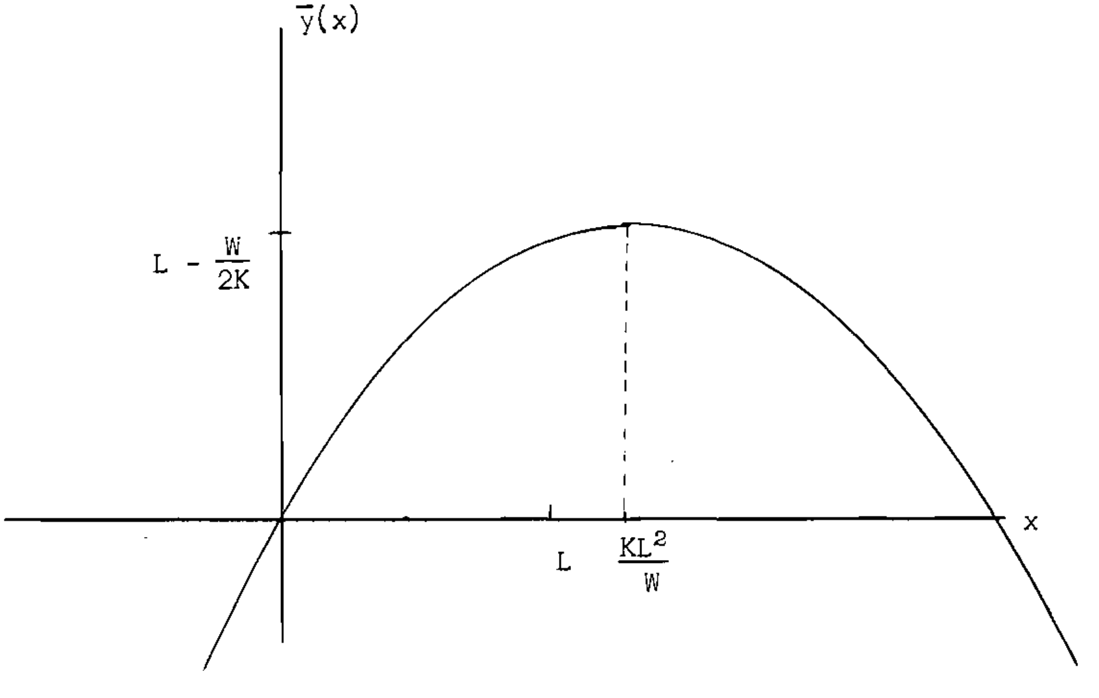

Figure 3. $\bar{y}(x)$ as a Function of x.

To insure that y, on the other hand, is one-to-one, and therefore that it has an inverse, it is necessary and sufficient to require that $L \leq \frac{KL^2}{W}$, or $\frac{W}{KL} \leq 1$. This stipulation has already been made. To insure that the inverse is differentiable throughout its domain, it is necessary to require that $\frac{W}{KL} < 1$ (otherwise, as Equation (15) shows, the linear density $\rho(y)$ becomes infinite at $y = L - \frac{W}{2K}$). This further restriction is therefore made. An explicit expression for $x(y)$ must now be found. If the equation

$$y = x - \frac{Wx^2}{2KL^2}$$

is solved for x, then two possibilities for x are confronted; namely,

$$x = \frac{1 + \sqrt{1 - 2Wy/KL^2}}{W/KL^2} \tag{12}$$

and

$$x = \frac{1 - \sqrt{1 - 2Wy/KL^2}}{W/KL^2} . \tag{13}$$

Since $x(0)$ must be equal to zero, Equation (13) must be chosen to represent $x(y)$. Hence,

$$x(y) = \frac{1 - \sqrt{1 - 2Wy/KL^2}}{W/KL^2} , \quad 0 \leq y \leq L - \frac{W}{2K} . \tag{14}$$

By combining the results of Equations (11) and (14), the desired result is obtained; that is,

$$\rho(y) = \frac{W/L}{\sqrt{1 - 2Wy/KL^2}} , \quad 0 \leq y \leq L - \frac{W}{2K} . \tag{15}$$

It is interesting to use Equation (15) to calculate the linear density at the lower end of the spring. Thus,

$$\rho(L - \frac{W}{2K}) = \frac{W/L}{\sqrt{1 - \frac{2W}{KL^2}(L - \frac{W}{2K})}} \tag{16}$$

$$= \frac{W/L}{\sqrt{(1 - W/KL)^2}}$$

$$= \frac{W/L}{|1 - W/KL|} \text{ .}$$

The requirement was made that $W/KL < 1$. Hence,

$$\rho(L - W/2K) = \frac{W/L}{1 - W/KL} \text{ .}$$

CHAPTER III

DYNAMIC SYSTEM WITH DISTRIBUTED MASS

Two masses m_1 and m_2 are joined together by a uniform linear spring of unstressed length L, mass M, and spring constant K. The whole configuration is allowed to fall through a frictionless vertical tube (Fig. 4). The distributed mass of the spring is not neglected, and it is not assumed that the spring is uniformly stretched.

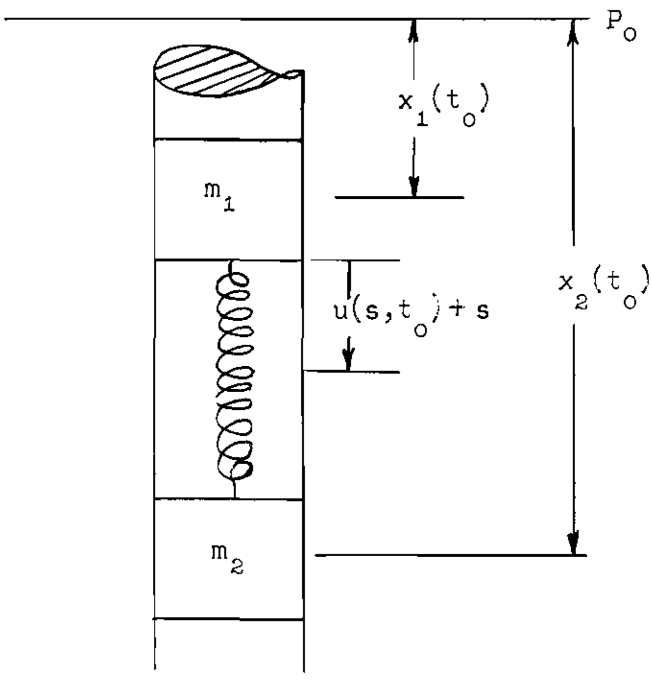

Figure 4. Position of the System at a Fixed Time

Let s: $0 \leq s \leq L$ denote the distance measured along the unstressed spring from m_1 toward m_2. When the system is in motion, let $u(s,t)$ denote for each fixed time $t \geq 0$ the displacement relative to the upper end of the spring of the material particle which in the unstressed state was s units from the upper end. It is assumed that $u \in C^2$ for $0 \leq s \leq L$ and $t \geq 0$, and it is required that $u(0,t) = 0$ for all $t \geq 0$, since the model is unrealistic unless the spring remains attached to the upper mass. Let P_o be a fixed horizontal plane, and let $x_1(t)$ and $x_2(t)$ be the distances of m_1 and m_2 measured positively downward from P_o. Suppose that x_1 and x_2 are twice-differentiable functions of t for $t \geq 0$. The "initial position" of the system referred to in the derivation is taken to be the state of the system in which the mass m_1 lies in the plane P_o and the spring is unstressed (Fig. 5).

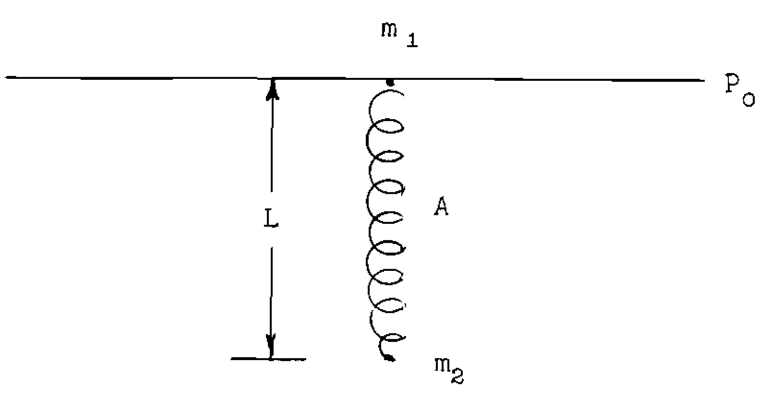

Figure 5. Initial State of the System

The mathematical model associated with this physical phenomenon consists of the equations of motion (with suitable boundary and initial conditions) for the system described above. Under the assumptions made, the application of Hamilton's Principle followed by the utilization of the methods of the calculus of variations gives an effective way to derive these equations. Thus, the integral

$$I = \int_0^{t'} (T - V)\, dt \qquad (t' > 0) ,$$

where T and V are the kinetic and potential energies of the system, must eventually be considered, but only after T and V have been found. The object of the present discussion is to derive expressions for these energies.

An expression for the kinetic energy T as a function of time is now derived. For $t \geq 0$ let $T_1(t)$ and $T_2(t)$ be the kinetic energies of m_1 and m_2, respectively; let $T_3(t)$ be the kinetic energy of the spring. The initial model-building step is the requirement that

$$T(t) = T_1(t) + T_2(t) + T_3(t), \qquad t \geq 0. \tag{17}$$

Now

$$T_1(t) = \frac{1}{2} m_1 [x_1'(t)]^2 , \tag{18}$$

and

$$T_2(t) = \frac{1}{2} m_2 [x_2'(t)]^2 . \tag{19}$$

The derivation of T_3 requires further analysis. Thus, a sequence $\{P_n\}$ of regular partitions of the interval $[0,L]$ is introduced, where

$P_n = \{0 = s_0^{(n)}, \; s_1^{(n)}, \; s_2^{(n)}, \ldots, s_{n-1}^{(n)}, \; s_n^{(n)} = L\}$ for $n = 1,2,$ $\ldots$. Let $h_i^{(n)} = s_i^{(n)} - s_{n-1}^{(n)} = \frac{L}{n}$. Note that $h_i^{(n)} \to 0$ as $n \to \infty$. For arbitrary fixed $t_o : t_o \geq 0$, let $T_3^{(i)}(t_o)$ denote the kinetic energy of that portion of the spring which in the initial state occupied the interval $[s_{i-1}^{(n)}, s_i^{(n)}]$ (see note below). It is required that

$$T_3(t_o) = \lim_{n \to \infty} \sum_{i=1}^{n} T_3^{(i)}(t_o) \, . \tag{20}$$

Consider the point which was at s on the spring in the initial position. For $t \geq 0$ the position of this point is $x_1(t) + u(s,t) + s$, $0 \leq s \leq L$. The function $u_t(s,t)$ is a continuous function of s for fixed t. Hence, the function $[x_1'(t_o) + u_t(s,t_o)]^2$ is a continuous function of s. Furthermore, if s is restricted to lie in the subinterval $[s_{i-1}^{(n)}, s_i^{(n)}]$, then by virtue of its continuity the function $[x_1'(t_o) + u_t(s,t_o)]^2$ assumes both an absolute maximum and an absolute minimum on the subinterval (Thereom 1, Appendix). The following inequality represents a physically motivated assertion and is a crucial assumption involved in this derivation.

$$\frac{M_i^{(n)}}{2} \operatorname*{Min}_s [x_1'(t_o) + u_t(s,t_o)]^2 \leq T_3^{(i)}(t_o)$$
$$\leq \frac{M_i^{(n)}}{2} \operatorname*{Max}_s [x_1'(t_o) + u_t(s,t_o)]^2 \, ,$$

Note. Although the dependence of $T_3^{(i)}(t_o)$ on n is not represented notationally, it should be remembered that it is present. This omission has been made to avoid cluttering the nomenclature.

where $s \in [s_{i-1}^{(n)}, s_i^{(n)}]$, and $M_i^{(n)}$ is the mass of that portion of the spring which initially occupied the i^{th} subinterval:

$$M_i^{(n)} = \frac{Mh_i^{(n)}}{L} .$$

Since the function $[x_1'(t_o) + u_t(s,t_o)]^2$ is continuous on $[s_{i-1}^{(n)}, s_i^{(n)}]$, it assumes every value between its absolute maximum and its absolute minimum on this subinterval (Theorem 2). It follows, then, that there exists a number $s_i^{(n)*} \in [s_{i-1}^{(n)}, s_i^{(n)}]$ such that

$$T_3^{(i)}(t_o) = \frac{Mh_i^{(n)}}{L}[x_1'(t_o) + u_t(s_i^{(n)*}, t_o)]^2 .$$

But this equation is valid for each $i = 1,2, \ldots, n$ and $n = 1,2, \ldots$. Hence, by substitution into Equation (20),

$$T_3(t_o) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{M}{2L}[x_1'(t_o) + u_t(s_i^{(n)*}, t_o)]^2 h_i^{(n)}. \qquad (21)$$

The function $[x_1'(t_o) + u_t(s,t_o)]^2$ is continuous and thus integrable on $[0,L]$. From Equation (21), the integrability implies that

$$T_3(t_o) = \frac{M}{2L}\int_0^L [x_1'(t_o) + u_t(s,t_o)]^2\, ds.$$

But $t_o \geq 0$ was arbitrary; therefore,

$$T_3(t) = \frac{M}{2L}\int_0^L [x_1'(t) + u_t(s,t)]^2\, ds, \qquad t \geq 0. \qquad (22)$$

From Equations (17), (18), (19), and (22) the total kinetic energy, $T(t)$, of the system can be found. Thus,

$$T(t) = \frac{1}{2} m_1 [x_1'(t)]^2 + \frac{1}{2} m_2 [x_2'(t)]^2$$

$$+ \frac{M}{2L} \int_0^L [x_1'(t) + u_t(s,t)]^2 \, ds, \qquad t \geq 0 .$$

The derivation of an expression for the potential energy V as a function of time is now considered. Potential energy is measured with respect to the state of the system in the initial position. For $t \geq 0$, let $V_1(t)$ and $V_2(t)$ denote the potential energies of m_1 and m_2; let $V_3(t)$ denote the potential energy of the spring due to the displacement of the spring in the gravitational field; let $V_4(t)$ be the energy stored in the spring due to its elongation. In the construction of the mathematical model the following assertion is made:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \qquad t \geq 0. \tag{23}$$

Now it can easily be seen that

$$V_1(t) = - m_1 \, g \, x_1(t) , \tag{24}$$

and

$$V_2(t) = - m_2 \, g[x_2(t) - L] . \tag{25}$$

Expressions for V_3 and V_4 must now be found. Refer to the sequence of partitions of the interval $[0,L]$ introduced in the derivation of T_3. Consider first V_3. For fixed time $t_1 \geq 0$, let $V_3^{(i)}(t_1)$ be the potential energy of that portion of the spring initially occupying the interval $[s_{i-1}^{(n)}, s_i^{(n)}]$. Then it is required that

$$V_3(t_1) = \lim_{n \to \infty} \sum_{i=1}^{n} V_3^{(i)} (t_1) . \tag{26}$$

By an argument similar to that which was expounded for T_3 the function $[x_1(t_1) + u(s,t_1)]$ is continuous and attains an absolute maximum and an absolute minimum on $[s_{i-1}^{(n)}, s_i^{(n)}]$ for fixed t_1. The following physically-motivated assumption is vital in the construction of the mathematical model.

$$- M_i^{(n)} g \operatorname*{Max}_s [x_1(t_1) + u(s,t_1)] \leq V_3^{(i)} (t_1)$$

$$\leq - M_i^{(n)} g \operatorname*{Min}_s [x_1(t_1) + u(s,t_1)],$$

where $s \in [s_{i-1}^{(n)}, s_i^{(n)}]$, and $M_i^{(n)}$ has the same meaning as in previous steps. In analogy to the calculation of T_3, an application of Theorem 2, Equation (26), and the continuity of $[x_1(t_1) + u(s,t_1)]$ imply that

$$V_3(t_1) = - \frac{Mg}{L} \int_0^L [x_1(t_1) + u(s,t_1)] \, ds$$

$$= - Mg x_1(t_1) - \frac{Mg}{L} \int_0^L u(s,t_1) \, ds .$$

Since $t_1 \geq 0$ was arbitrarily chosen, for every $t \geq 0$,

$$V_3(t) = - Mg x_1(t) - \frac{Mg}{L} \int_0^L u(s,t) \, ds . \tag{27}$$

In order to derive an expression for V_4, the element of unstressed length $h_i^{(n)}$ occupying the interval $[s_{i-1}^{(n)}, s_i^{(n)}]$ in the unstressed state is again considered. If this element were stretched uniformly, then the potential energy stored in it due to stretching would be $\frac{1}{2}(LK/h_i^{(n)})[u(s_i^{(n)}, t_2) - u(s_{i-1}^{(n)}, t_2)]^2$ for fixed $t_2 \geq 0$. But by Theorem 3,

$$u(s_i^{(n)}, t_2) - u(s_{i-1}^{(n)}, t_2) = u_s(\zeta, t_2)\, h_i^{(n)}, \qquad s_{i-1}^{(n)} < \zeta < s_i^{(n)},$$

where, since the stretching is uniform, $u_s(\zeta, t_2)$ has the same value for each arbitrarily chosen $\zeta \varepsilon (s_{i-1}^{(n)}, s_i^{(n)})$. Hence, for uniform stretching the expression for potential energy becomes $\frac{1}{2} LKh_i^{(n)} u_s^2(s, t_2)$. Now suppose that the element is not uniformly stretched. Then $u_s(s, t_2)$ is no longer independent of s. It is asserted that the energy stored in the element $[s_{i-1}^{(n)}, s_i^{(n)}]$ will lie between the following bounds:

(a) the energy which would be stored in a similar element uniformly stretched in such a way that its elongation is $\underset{s}{\text{Max}}\, |u_s(s, t_2)|\, h_i^{(n)}$;

(b) the energy which would be stored in a similar element uniformly stretched in such a way that its elongation is $\underset{s}{\text{Min}}\, |u_s(s, t_2)|\, h_i^{(n)}$,

where $s \varepsilon [s_{i-1}^{(n)}, s_i^{(n)}]$. That is, upper and lower bounds of the energy stored in the actual, nonuniformly-stretched element are respectively $\frac{1}{2}(LK/h_i^{(n)})\, [h_i^{(n)} \underset{s}{\text{Max}}\, |u_s(s, t_2)|]^2$ and $\frac{1}{2}(LK/h_i^{(n)})[h_i^{(n)} \underset{s}{\text{Min}}$ $|u_s(s, t_2)|]^2$, for fixed $t_2 \geq 0$ and $s \varepsilon [s_{i-1}^{(n)}, s_i^{(n)}]$. Thus the following requirement is made, where $V_4^{(i)}$ is the energy stored in the element $[s_{i-1}^{(n)}, s_i^{(n)}]$:

$$\frac{1}{2}\frac{LK}{h_i^{(n)}}[h_i^{(n)} \operatorname*{Min}_s |u_s(s,t_2)|]^2 \leq V_4^{(i)}$$

$$\leq \frac{1}{2}\frac{LK}{h_i^{(n)}}[h_i^{(n)} \operatorname*{Max}_s |u_s(s,t_2)|]^2 ,$$

for $s \varepsilon [s_{i-1}^{(n)}, s_i^{(n)}]$. By an argument analogous to those for T_3 and V_3 the following result is obtained:

$$V_4(t) = \frac{KL}{2}\int_0^L [u_s(s,t)]^2\, ds, \qquad t \geq 0. \tag{28}$$

By using Equations (23), (24), (25), (27), and (28) the expression for V can be constructed. Thus,

$$V(t) = -m_1 g x_1(t) - m_2 g[x_1(t) - L] - Mgx_1(t)$$

$$-\frac{Mg}{L}\int_0^L u(s,t)\, ds + \frac{1}{2}KL\int_0^L [u_s(s,t)]^2\, ds ,$$

for $t \geq 0$.

Though not the primary object of this discussion, it may be of some interest to note that application of Hamilton's Principle and the methods of the calculus of variations lead to the following equations of motion for the system:

$$Mu_{tt}(s,t) - KL^2 u_{ss}(s,t) = M[g - x_1''(t)];$$

$$m_1\, x_1''(t) = m_1\, g + KL\, u_s(0,t);$$

$$m_2\, x_2''(t) = m_2\, g - KL\, u_s(L,t).$$

CHAPTER IV

DYNAMICAL EQUATIONS FOR A PERFECT FLUID

Let A be a simply-connected region in E_3 containing in its interior a closed rectangular parallelepiped R of positive dimensions h, k, and l with one corner at an arbitrary point (x_o, y_o, z_o) (Fig. 6). Suppose that A is pervaded by perfect fluid of density ρ. The dynamical equations of the fluid in A are to be found, and the object of this chapter is to illustrate how a higher-dimensional law of the mean is used in that process.

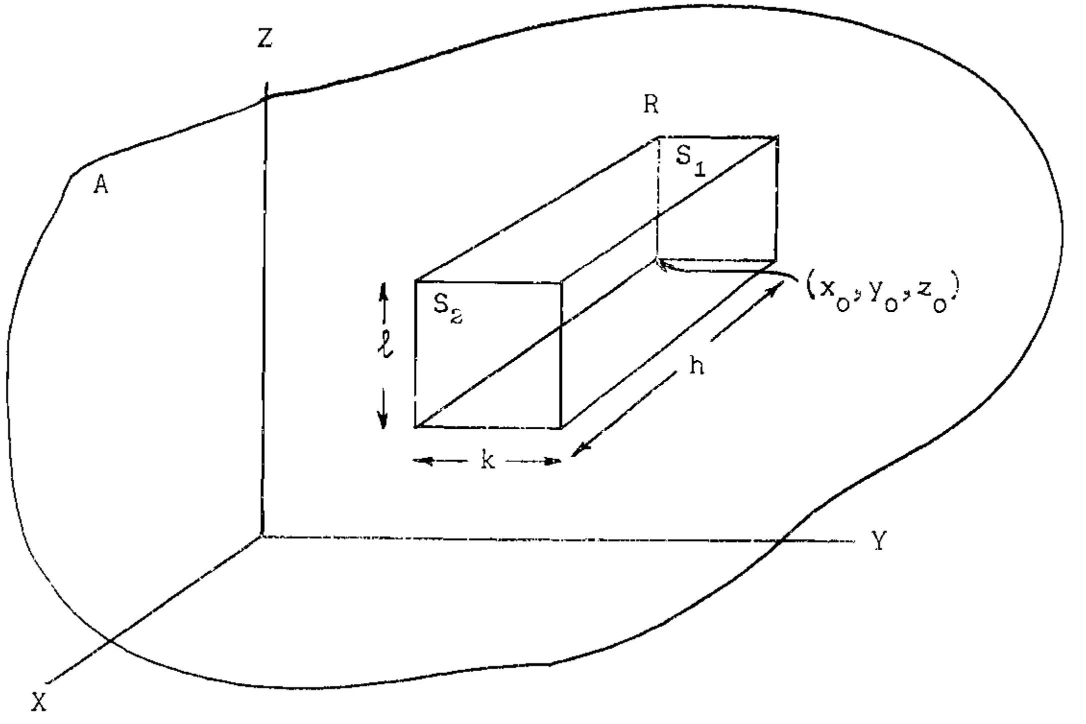

Figure 6. Rectangular Parallelepiped R in A

Let $p(x,y,z,t)$ denote for each fixed time $t \geq 0$ the pressure at $(x,y,z) \varepsilon A$. It is assumed that $p \varepsilon C^1$ in A for fixed $t \geq 0$. The face of R parallel to the YZ-plane and containing the point $(x_o + h, y_o, z_o)$ is denoted by S_2, and the face of R parallel to the YZ-plane and containing the point (x_o, y_o, z_o) is denoted by S_1. Consider S_1, a set on which p is continuous. By Theorem 4 and Theorem 5 the average value of p on S_1, denoted by $\overline{p}$, is given by the equation

$$\overline{p} = \frac{1}{k\,l} \int_{S_1}\int p(x_o, y, z, t_o)\, dA , \tag{29}$$

where t_o is fixed and nonnegative. Similarly, the average value of p on S_2, denoted by $\overline{\overline{p}}$, is given by the equation

$$\overline{\overline{p}} = \frac{1}{k\,l} \int_{S_2}\int p(x_o + h, y, z, t_o)\, dA . \tag{30}$$

The net force in the positive X-direction due to the difference of pressure on the two faces is given by

$$\tilde{F}_1 = (\overline{p} - \overline{\overline{p}})\, kl . \tag{31}$$

Consider the difference $\overline{p} - \overline{\overline{p}}$. According to Equations (29) and (30)

$$p - \overline{\overline{p}} = \frac{1}{k\,l} \left\{ \int_{S_1}\int p(x_o, y, z, t_o)\, dA \right.$$

$$\left. - \int_{S_2}\int p(x_o + h, y, z, t_o)\, dA \right\} .$$

But when considered as sets of ordered pairs (y,z), S_1 and S_2 are identical; so,

$$\overline{p} - \overline{\overline{p}} = \frac{1}{kl} \iint_{S_1} [p(x_o,y,z,t_o) - p(x_o + h, y,z,t_o)]dA. \tag{32}$$

Since p is a continuous function of its first argument, by a mean value theorem there exists at least one function $\theta(y,z)$ such that for $(y,z) \in S_1$, $0 < \theta(y,z) < 1$, and

$$p(x_o,y,z,t_o) - p(x_o + h, y,z,t_o) = - p_x(x_o + h\theta(y,z), y,z,t_o)\, h. \tag{33}$$

From Equations (32) and (33) the following equation arises:

$$\overline{p} - \overline{\overline{p}} = \frac{-h}{kl} \iint_{S_1} p_x(x_o + h\theta(y,z), y,z,t_o)\, dA. \tag{34}$$

If it can be shown that p_x is a continuous function of (y,z) on S_1, then by Theorem 5, the equation

$$\iint_{S_1} p_x(x_o + h\theta(y,z), y,z,t_o)\, dA \tag{35}$$

$$= p_x(x_o + h\theta(\overline{y},\overline{z}), \overline{y},\overline{z},t_o)\, kl$$

will follow, where $(\overline{y},\overline{z}) \in S_1$. This continuity is now demonstrated. Let (y',z') and (y'',z'') represent points in S_1. It must be proved that for each $\varepsilon > 0$ there exists a positive number δ (which may depend on ε, x_o, h, y', y'', z', z'', and t_o) such that

$$|p_x(x_o + h\theta(y',z'), y',z',t_o) - p_x(x_o + h\theta(y'',z''), y'',z'',t_o)| < \varepsilon$$

whenever $0 < |(y',z') - (y'',z'')| < \delta$. Let $\varepsilon > 0$ be given. Now by (33)

$$|p_x(x_o + h\theta(y',z'),y',z',t_o) - p_x(x_o + h\theta(y'',z''),y'',z'',t_o)|$$

$$= |[p(x_o,y',z',t_o) - p(x_o + h,y',z',t_o)]$$

$$- [p(x_o,y'',z'',t_o) - p(x_o + h,y'',z'',t_o)]| .$$

A slight rearrangement, followed by application of a triangle inequality, yields

$$|p_x(x_o + h\theta(y',z'),y',z',t_o) - p_x(x_o + h\theta(y'',z''),y'',z'',t_o)|$$

$$= |[p(x_o,y',z',t_o) - p(x_o,y'',z'',t_o)]$$

$$- [p(x_o + h,y',z',t_o) - p(x_o + h,y'',z'',t_o)]|$$

$$\leq |p(x_o,y',z',t_o) - p(x_o,y'',z'',t_o)| \tag{36}$$

$$+ |p(x_o + h,y',z',t_o) - p(x_o + h,y'',z'',t_o)|.$$

Now p is a continuous function of (y,z) on S_1. Hence, for each $\varepsilon > 0$ there exist positive numbers δ_1 and δ_2 (which may depend on ε, x_o,y',y'',z',z'',t_o) such that $|p(x_o,y',z',t_o) - p(x_o,y'',z'',t_o)| < \frac{\varepsilon}{2}$ whenever $0 < |(y',z') - (y'',z'')| < \delta_1$, and $|p(x_o + h,y',z',t_o) - p(x_o + h, y'',z'',t_o)| < \frac{\varepsilon}{2}$ whenever $0 < |(y',z') - y'',z'')| < \delta_2$. Let $\delta = \text{Min}[\delta_1,\delta_2]$. Then from inequality (36) it follows that

$$|p_x(x_o + h\theta(y',z'),y',z',t_o) - p_x(x_o + h\theta(y'',z''),y'',z'',t_o)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever (y',z') and (y'',z'') are in S_1 and $0 < |(y',z') - (y'',z'')| < \delta$. Hence, $p_x(x_o + h\theta(y,z),y,z,t_o)$ is a continuous function of (y,z) on S_1.

Therefore, Equation (35) holds, and by combining Equations (34) and (35) the following result is obtained:

$$\overline{p} - \overline{\overline{p}} = - h\, p_x(x_o + h\theta(\overline{y},\overline{z}),\overline{y},\overline{z},t_o) ,$$

where $(\overline{y},\overline{z}) \varepsilon S_1$. From Equation (31) the net force in the X-direction is given by

$$\tilde{F}_1 = - h\,k\,l\, p_x(x_o + h\theta(\overline{y},\overline{z}),\overline{y},\overline{z},t_o) , \tag{37}$$

for some $(\overline{y},\overline{z}) \varepsilon S_1$.

Possible external forces must now be considered. Let $\vec{G}(x,y,z,t)$ denote for each fixed $t \geq 0$ the external force per unit volume exerted at $(x,y,z) \varepsilon R$. It is assumed that each of the components of $\vec{G}$ is continuous in (x,y,z) for fixed nonnegative t. Consider the X-component of $\vec{G}$, denoted by G_1. Since G_1 is continuous on the compact connected set R, by Theorem 4 there exist numbers θ_o', θ_1', and θ_2' (each between 0 and 1) such that $\tilde{G}_1$, the net external force in the X-direction, is given by the equation

$$\tilde{G}_1 = G_1(x_o + \theta_o' h, y_o + \theta_1' k, z_o + \theta_2' l, t_o)\, h\, k\, l\,. \tag{38}$$

The X-component of the total force on R is $\tilde{F}_1 + \tilde{G}_1$. By Newton's Second Law this sum is equal to an inertial reaction $\tilde{I}_1$ given by the equation

$$\tilde{I}_1 = a_1(x_o + \theta_o'' h, y_o + \theta_1'' k, z_o + \theta_2'' l, t_o)\, \rho\, h\, k\, l\,, \tag{39}$$

where a_1 is the X-component of the acceleration $\vec{a}$ of the particles of fluid in R. The function a_1 is assumed continuous in R, and this stipulation implies the existence of numbers θ_o'', θ_1'', and θ_2'' (between 0 and 1) which give a_1 the appropriate value according to Theorem 4.

From Equations (37), (38), and (39) the following equation is obtained by application of Newton's Second Law:

$$\rho\, h\, k\, l\, a_1(x_o + \theta_o'' h, y_o + \theta_1'' k, z_o + \theta_2'' l, t_o)$$

$$= h\, k\, l\, [-p_x(x_o + h\theta(\bar{y},\bar{z}), \bar{y}, \bar{z}, t_o)$$

$$+ G_1(x_o + \theta_o' h, y_o + \theta_1' k, z_o + \theta_2' l, t_o)]\,.$$

If this last equation is divided by the positive quantity hkl, the resulting equation is

$$\rho\, a_1(x_o + \theta_o'' h, y_o + \theta_1'' k, z_o + \theta_2'' l, t_o)$$

$$= G_1(x_o + \theta_o' h, y_o + \theta_1' k, z_o + \theta_2' l, t_o) -$$

$$- p_x(x_o + h\theta(\overline{y},\overline{z}),\overline{y},\overline{z},t_o).* \tag{40}$$

The limits of both sides of Equation (40) as $(h,k,l) \to (0^+,0^+,0^+)$ now are considered. Since the functions a_1, G_1, and p_x are all continuous and $0 < \theta(\overline{y},\overline{z}) < 1$, these limits exist and are equal to each other. The equation resulting from the limit is

$$\rho\, a_1(x_o,y_o,z_o,t_o) = G_1(x_o,y_o,z_o,t_o) - p_x(x_o,y_o,z_o,t_o).$$

But (x_o,y_o,z_o) is an arbitrarily chosen point in A, and t_o is an arbitrary nonnegative time. Hence, for all $(x,y,z) \varepsilon A$ and $t \geq 0$,

$$\rho\, a_1(x,y,z,t) = G_1(x,y,z,t) - p_x(x,y,z,t). \tag{41}$$

If the preceding argument were carried through for the Y- and Z-components, then equations similar to Equation (41) would be obtained. Thus, the dynamical equations for the fluid in A can be given by the single equation

$$\rho\, \vec{a}(x,y,z,t) = \vec{G}(x,y,z,t) - \vec{\nabla} p\,(x,y,z,t),$$

for all $(x,y,z) \varepsilon A$, and $t \geq 0$.

---

*Note that $\overline{y}$ and $\overline{z}$ are dependent upon k and l, respectively. That is, $\overline{y}$ and $\overline{z}$ can be written as $y_o + k\theta_1$ and $z_o + l\theta_2$, where $0 \leq \theta_1 \leq 1$, and $0 \leq \theta_2 \leq 1$. Thus $\overline{y}$ and $\overline{z}$ approach y_o and z_o, respectively, as k and l approach zero.

CHAPTER V

SOME WELL-KNOWN DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

Three well-known partial differential equations of physics are the wave equation, the heat equation, and Laplace's equation. Each of these equations arises as the mathematical model associated with several types of physical phenomena. The purpose of this chapter is to present careful derivations of these equations. The physical model in each case is simple and easily visualized, thus allowing emphasis to be placed on the mathematical aspects of the derivation.

Displacements in a Flexible String with External Force Applied

Consider a flexible string of length L initially occupying the interval $[0,L]$ of the X-axis, which is taken to be horizontal. Let $u(x,t)$ represent the vertical displacement at time t of the point which was initially at $x \varepsilon [0,L]$ (Fig. 7). It is assumed that $u \varepsilon C^2$ for $0 \leq x \leq L$ and $t \geq 0$. Let $f(x,t)$ represent at time t the external force per unit length applied in a vertical direction at the point which was originally at $x \varepsilon [0,L]$, where $f \varepsilon C$ for $0 \leq x \leq L$ and $t \geq 0$. It is assumed that $f(x,t)$ is independent of $u(x,t)$; and in addition, the following physical assumptions are made regarding the string.

(1) The string is flexible to the extent that all bending moments transmitted between its elements can be neglected; hence, it transmits only tangential forces.

I.

II.

III.

IV.

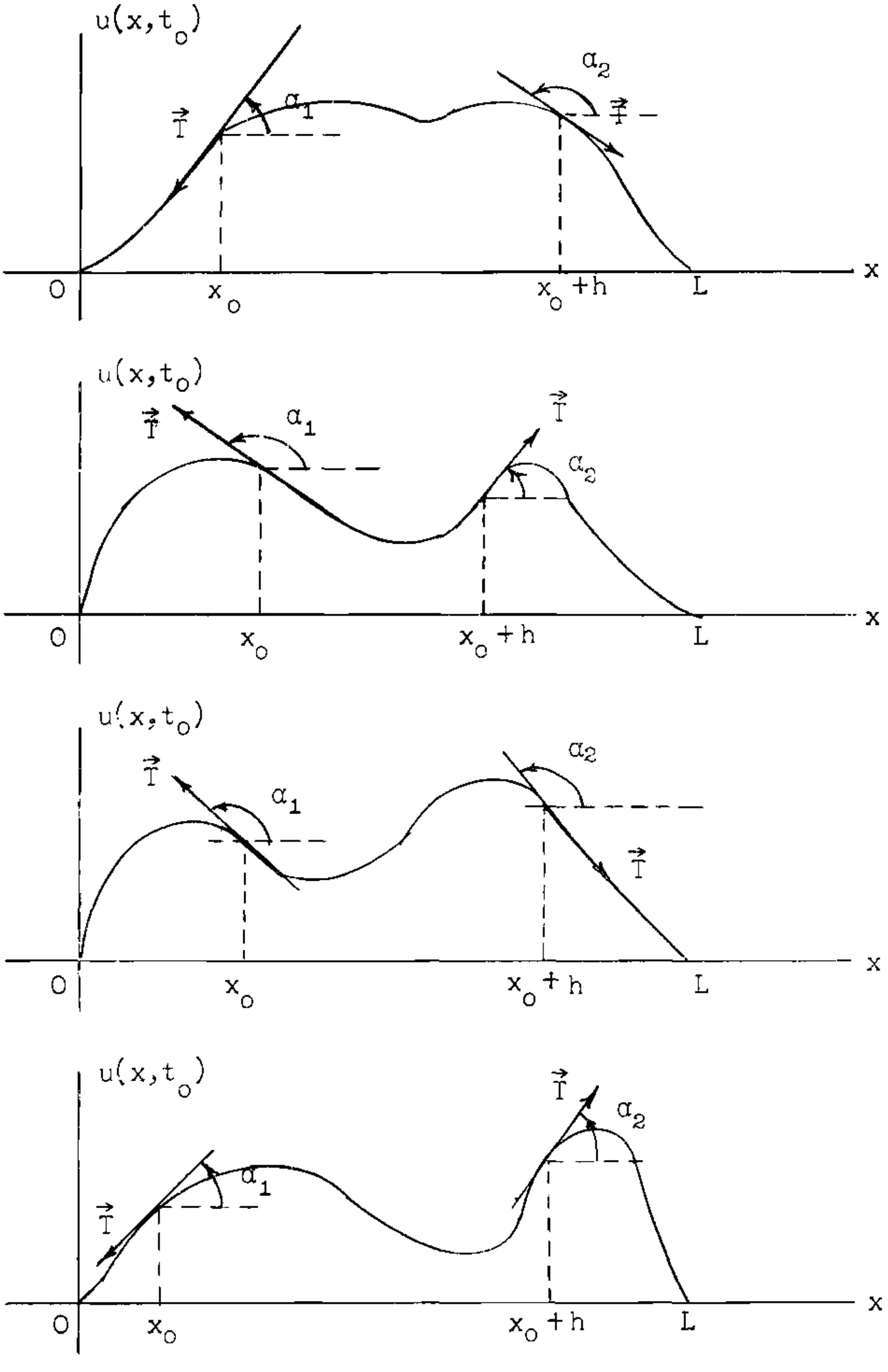

Figure 7. Positions of the Vibrating String

(2) A tension $\vec{T}$ of constant magnitude $T(T > 0)$ is present throughout the length of the string.

(3) The displacement $u(x,t)$ is small compared to the length of the string for all $x \varepsilon [0,L]$ and $t \geq 0$. Furthermore $u_x^2(x,t)$ is small compared to unity for these same values of x and t.

(4) Only vertical displacement takes place for every point of the string for $t \geq 0$.*

The object of the problem is to deduce, under the assumptions above, a differential equation involving u as the dependent variable. The differential equation, together with appropriate boundary and initial conditions, constitutes the mathematical model desired.

Let $t_o \geq 0$ be fixed and consider a segment of the string initially occupying the interval $[x_o, x_o + h]$ contained in $[0,L]$, where $h > 0$ (the case for $h < 0$ is entirely analogous). An expression for the total vertical force on this segment at time t_o is to be found. Consider the tension $\vec{T}$. Let α_1 and α_2 be the angles of inclination of the tangents at the points $(x_o, u(x_o, t_o))$ and $(x_o + h, u(x_o + h, t_o))$, respectively (Fig. 7). Then the vertical component of the tension at x_o is $-T(\sin \alpha_1)[\operatorname{sgn} u_x(x_o,t_o)]$ and at $x_o + h$ is $T(\sin \alpha_2)[\operatorname{sgn} u_x(x_o + h, t_o)]$. Consider now the total vertical external force on the segment. It is denoted by F. Since $f \varepsilon C$ on

*Note that this assumption has the following implication: for each arbitrary interval $[x_1, x_2]$, the mass of that part of the vibrating string which lies in or above the interval is the same as the mass of that part of the string which occupied the interval $[x_1, x_2]$ when the string was in the neutral position.

$[x_o, x_o + h]$, this function assumes both an absolute maximum and an absolute minimum for $x \varepsilon [x_o, x_o + h]$ (Theorem 1). The following assertion is made:

$$h \operatorname*{Min}_{x} [f(x,t_o)] \leq F \leq h \operatorname*{Max}_{x} [f(x,t_o)] ,$$

where $x \varepsilon [x_o, x_o + h]$. Since f is a continuous function of x alone on $[x_o, x_o + h]$, then by Theorem 3 there is at least one number θ_1: $0 < \theta_1 < 1$ such that

$$F = h\, f(x_o + \theta_1 h, t_o) .$$

Hence, the total force exerted on the segment initially occupying the interval $[x_o, x_o + h]$ is

$$T(\sin \alpha_2)[\operatorname{sgn} u_x(x_o + h, t_o)] - T(\sin \alpha_1)[\operatorname{sgn} u_x(x_o, t_o)]$$
$$+ h\, f(x_o + \theta_1 h, t_o) .$$

By Newton's Second Law this force must be equal to some inertial reaction I_F. To facilitate the derivation of an expression for I_F the following assertion is made, where $x \varepsilon [x_o, x_o + h]$:

$$\rho\, h \operatorname*{Min}_{x} [u_{tt}(x,t_o)] \leq I_F \leq \rho\, h \operatorname*{Max}_{x} [u_{tt}(x,t_o)],$$

for u_{tt} is continuous in x on $[x_o, x_o + h]$ and thus assumes both an absolute maximum and an absolute minimum on this interval. Here ρ is mass per unit length of the unstressed string. By application of Theorem 3 there exists a θ_2: $0 < \theta_2 < 1$ such that

$$I_F = \rho h u_{tt}(x_o + \theta_2 h, t_o).$$

From Newton's Second Law the equation

$$\rho h u_{tt}(x_o + \theta_2 h, t_o) = T(\sin \alpha_2)[\operatorname{sgn} u_x(x_o + h, t_o)] - T(\sin \alpha_1)[\operatorname{sgn} u_x(x_o, t_o)] + h f(x_o + \theta_1 h, t_o) \tag{42}$$

then follows. But

$$\sin \alpha_1 = \frac{(\tan \alpha_1)[\operatorname{sgn} \tan \alpha_1]}{\sqrt{1 + \tan^2 \alpha_1}}$$

$$= \frac{u_x(x_o, t_o)[\operatorname{sgn} u_x(x_o, t_o)]}{\sqrt{1 + u_x^2(x_o, t_o)}}; \tag{43}$$

and similarly,

$$\sin \alpha_2 = \frac{u_x(x_o + h, t_o)[\operatorname{sgn} u_x(x_o + h, t_o)]}{\sqrt{1 + u_x^2(x_o + h, t_o)}}. \tag{44}$$

Substituting the results of Equations (43) and (44) into Equation (42) gives the equation

$$\rho h u_{tt}(x_o + \theta_2 h, t_o) = \frac{T u_x(x_o + h, t_o)[\operatorname{sgn} u_x(x_o + h, t_o)]^2}{\sqrt{1 + u_x^2(x_o + h, t_o)}} - \frac{T u_x(x_o, t_o)[\operatorname{sgn} u_x(x_o, t_o)]^2}{\sqrt{1 + u_x^2(x_o, t_o)}} + h f(x_o + \theta_1 h, t_o),$$

which, when simplified, becomes

$$\rho h u_{tt}(x_o + \theta_2 h, t_o) = T \left\{ \frac{u_x(x_o + h, t_o)}{\sqrt{1 + u_x^2(x_o + h, t_o)}} \right. \tag{45}$$

$$\left. - \frac{u_x(x_o, t_o)}{\sqrt{1 + u_x^2(x_o, t_o)}} \right\} + h f(x_o + \theta_1 h, t_o) \, .$$

The function $u_x(x, t_o)/\sqrt{1 + u_x^2(x, t_o)}$, however, is a continuous function of x on $[x_o, x_o + h]$ and a differentiable function of x in $(x_o, x_o + h)$. Hence, by Theorem 3 there exists a number θ_3: $0 < \theta_3 < 1$ such that the difference in the brackets in Equation (45) can be written in the form

$$\frac{u_x(x_o + h, t_o)}{\sqrt{1 + u_x^2(x_o + h, t_o)}} - \frac{u_x(x_o, t_o)}{\sqrt{1 + u_x^2(x_o, t_o)}}$$

$$= h \left. \left\{ \frac{d}{dx} \left[\frac{u_x(x, t_o)}{\sqrt{1 + u_x^2(x, t_o)}} \right] \right\} \right|_{x = x_o + \theta_3 h}$$

Upon performing the indicated differentiation and evaluation, the following equation for the difference is obtained:

$$\frac{u_x(x_o + h, t_o)}{\sqrt{1 + u_x^2(x_o + h, t_o)}} - \frac{u_x(x_o, t_o)}{\sqrt{1 + u_x^2(x_o, t_o)}} \tag{46}$$

$$= h \, \frac{u_{xx}(x_o + \theta_3 h, t_o)}{[1 + u_x^2(x_o + \theta_3 h, t_o)]^{3/2}} \, .$$

Thus, from Equations (45) and (46) it follows that

$$\rho h u_{tt}(x_o + \theta_2 h, t_o) = hT \frac{u_{xx}(x_o + \theta_3 h, t_o)}{[1 + u_x^2(x_o + \theta_3 h, t_o)]^{3/2}} \tag{47}$$

$$+ \; h f(x_o + \theta_1 h, t_o) \, .$$

Equation (47) is divided by h; and by using the continuity requirements on the functions u and f, the limit as $h \to 0^+$ is found to be

$$\rho u_{tt}(x_o, t_o) = \frac{T u_{xx}(x_o, t_o)}{[1 + u_x^2(x_o, t_o)]^{3/2}} + f(x_o, t_o) \, . \tag{48}$$

It can easily be shown that Equation (48) is also obtained from a similar expression in which $h < 0$ by considering the limit as $h \to 0^-$. Hence, the limit as $h \to 0$ exists and Equation (48) follows as a result. But x_o and t_o are arbitrary. Therefore, for all $x \in [0,L]$ and $t \geq 0$,

$$\rho u_{tt}(x,t) = \frac{T u_{xx}(x,t)}{[1 + u_x^2(x,t)]^{3/2}} + f(x,t) \, . \tag{49}$$

Equation (49) is a nonlinear partial differential equation which, however, must be linearized in order to be consistent with the assumption (3) that $1 + u_x^2 \approx 1$. Thus, for the linear version

$$\rho u_{tt}(x,t) = T u_{xx}(x,t) + f(x,t) \, , \tag{50}$$

the standard form of the wave equation in one space dimension with external force applied.

As a final observation, attention is invited to the fact that assumptions (2) and (4) together with Newton's Second Law lead to a contradiction unless assumption (3) is also used. Consider a segment of the moving string (Fig. 7), and sum the X-components of the forces acting on the segment. Since only vertical displacements are assumed to occur, the inertial reaction has no X-component. Furthermore, the applied force $f(x,t)$ is vertically directed. Hence, in the nomenclature previously used,

$$-T\ (\cos\alpha_1)[\operatorname{sgn}(\cos\alpha_1)] + T\ (\cos\alpha_2)[\operatorname{sgn}(\cos\alpha_2)] = 0\ .$$

But

$$\cos\alpha_1 = \frac{\operatorname{sgn} u_x(x_o,t_o)}{\sqrt{1 + u_x^2(x_o,t_o)}}\ ,$$

and

$$\cos\alpha_2 = \frac{\operatorname{sgn} u_x(x_o + h,t_o)}{\sqrt{1 + u_x^2(x_o + h,t_o)}}\ .$$

So

$$T\left(\frac{1}{\sqrt{1 + u_x^2(x_o,t_o)}} - \frac{1}{\sqrt{1 + u_x^2(x_o + h,\ t_o)}}\right) = 0\ ,$$

or, since $T \neq 0$,

$$\frac{1}{\sqrt{1 + u_x^2(x_o + h,t_o)}} - \frac{1}{\sqrt{1 + u_x^2(x_o,t_o)}} = 0\ ,$$

a condition not satisfied in general since $u_x^2(x_o + h,t_o)$ is not identically equal to $u_x^2(x_o,t_o)$ in h for arbitrary x_o and t_o. The contradiction is eliminated, however, when assumption (3) is adduced, and $u_x^2(x_0,t_0)$ is neglected in comparison with unity. In the light of these remarks, and contrary to an often-held view, one might reasonably ask whether the nonlinear differential equation (49) represents the physical system any more faithfully than does the linear equation (50); for until the linearizing approximation is made, the contradiction just outlined is present.

Linear Flow of Heat Through a Cylindrical Rod

Consider a right cylindrical rod of length L and cross-sectional area A. Suppose

(1) that the material of which the rod is composed has thermal conductivity K, density ρ, and specific heat c, all of which are time-independent but may depend in a continuously differentiable manner upon distance along the rod;

(2) that the ends of the rod $(x = 0$ and $x = L)$ (Fig. 8) are maintained at constant temperatures u_0 and u_L, respectively, where $u_0 > u_L$;

(3) that there are no sources nor sinks of heat in the interior of the rod;

(4) that the lateral surface of the rod is completely insulated;

(5) that changes in the dimensions of the rod with temperature are small.

Let $u(x,t)$ be the measure of the temperature at time t: $t \geq 0$ in the cross-section x units from the left end of the rod, where $0 \leq x \leq L$. Furthermore, assume that $u \in C^2$ for $0 \leq x \leq L$, $t \geq 0$.

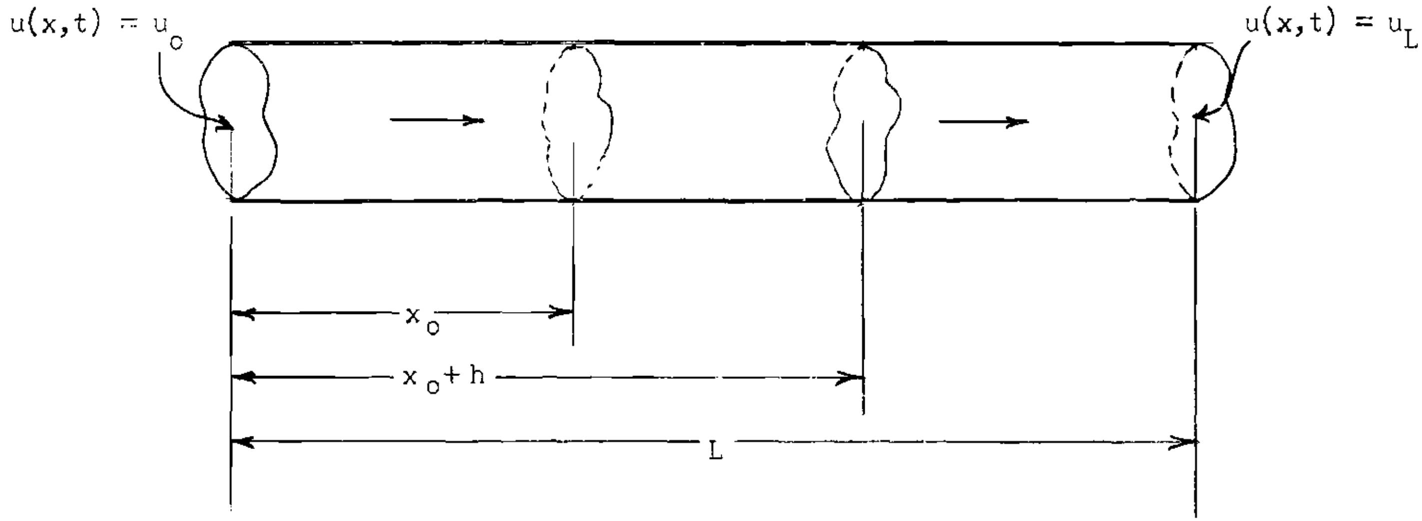

Figure 8. Cylindrical Rod

Note that the propriety of writing $u(x,t)$ as a function of only one geometric coordinate depends on the constancy of the temperature over a cross-section for fixed t. In the present discussion this constancy is assumed in the process of constructing the mathematical model. To show analytically that it is in fact a consequence of the other assumptions would in itself be a nontrivial mathematical problem.

The object of this discussion is to derive, within the bounds of the assumptions outlined above, a partial differential equation with u as the dependent variable to which may be added appropriate boundary and initial conditions. The differential equation, together with the extra conditions, will constitute a presumably accurate mathematical model descriptive of the linear flow of heat in the rod.

For fixed $x_o \in [0,L]$ consider the portion of the rod for which $x_o \leq x \leq x_o + h$, where $h > 0$. The rate at which heat is entering this volume through the plane $x = x_o$ at fixed time t_o is $-AK(x_o)u_x(x_o,t_o)$. Similarly, the rate at which heat is leaving the volume at the same instant through the plane $x = x_o + h$ is $-AK(x_o + h)\, u_x(x_o + h, t_o)$. From assumptions (3) and (4) it follows that no heat is either generated or lost within the volume, and none escapes or enters through the lateral sides. Hence, the difference between the rate at which heat enters and the rate at which heat leaves is the rate of increase of heat energy stored in the volume. This rate of increase is denoted by $Q_t(x_o,h,t_o)$. Thus,

$$Q_t(x_o,h,t_o) = -AK(x_o)\, u_x(x_o,t_o) - [-AK(x_o + h)\, u_x(x_o + h,t_o)] ,$$

or

$$Q_t(x_o,h,t_o) \;=\; A\,[K(x_o + h)\; u_x(x_o + h,t_o) - K(x_o)\; u_x(x_o,t_o)]. \qquad (51)$$

But since the product function $(Ku_x) \varepsilon C^1$ on $[x_o, x_o + h]$ for fixed t_o, there exists a number θ_1: $0 < \theta_1 < 1$ such that

$$K(x_o + h)\; u_x(x_o + h,t_o) - K(x_o)\; u_x(x_o,t_o) \qquad (52)$$

$$= h\left\{\frac{\partial}{\partial x}[K(x)\; u_x(x,t_o)]\right\}\Bigg|_{x = x_o + \theta_1 h} .$$

Therefore, for fixed x_o and t_o, Equations (51) and (52) imply the equation

$$Q_t(x_o,h,t_o) \;=\; A\,h\left\{\frac{\partial}{\partial x}[K(x)\; u_x(x,t_o)]\right\}\Bigg|_{x = x_o + \theta_1 h} . \qquad (53)$$

The rate of increase of heat stored in the volume bounded by the surface of the rod and the planes $x = x_o$ and $x = x_o + h$ can also be expressed in terms of the rate of rise of temperature within this volume. To make this statement precise, however, either something must be said about the point in the volume where the rate of rise of temperature is determined, or it must be shown that the location of this point is immaterial. A little reflection shows, in fact, that a familiar assertion may be made:

$$A\,h \underset{x_o \leq x \leq x_o + h}{\text{Minimum}} [\rho(x)\; c(x)\; u_t(x,t_o)] \;\leq\; Q_t(x_o,h,t_o)$$

$$\leq\; A\,h \underset{x_o \leq x \leq x_o + h}{\text{Maximum}} [\rho(x)\; c(x)\; u_t(x,t_o)],$$

since it is assured by the continuity of $(\rho c u_t)$ that this function assumes both an absolute maximum and an absolute minimum for $x \varepsilon$ $[x_o, x_o + h]$ and fixed $t_o \geq 0$. This continuity in connection with the preceding inequality furthermore guarantees the existence of a number θ_2: $0 < \theta_2 < 1$ such that

$$Q_t(x_o, h, t_o) = A\,h\,\rho(x_o + \theta_2 h)\; c(x_o + \theta_2 h)\; u_t(x_o + \theta_2 h, t_o) . \tag{54}$$

Equations (53) and (54) when combined yield the following equation:

$$A\,h \left\{ \frac{\partial}{\partial x} [K(x)\; u_x(x, t_o)] \right\} \bigg|_{x = x_o + \theta_1 h}$$

$$= A\,h\,\rho(x_o + \theta_2 h)\; c(x_o + \theta_2 h)\; u_t(x_o + \theta_2 h, t_o) .$$

Division by $A\,h$ gives

$$\left\{ \frac{\partial}{\partial x} [K(x)\; u_x(x, t_o)] \right\} \bigg|_{x = x_o + \theta_1 h} \tag{55}$$

$$= \rho(x_o + \theta_2 h)\; c(x_o + \theta_2 h)\; u_t(x_o + \theta_2 h, t_o) .$$

A result identical to Equation (55) is obtained under the assumption that $h < 0$. Hence, the limit as $h \to 0$ of both sides of this equation may be considered. By utilizing the continuity of the functions $\frac{\partial}{\partial x} [K(x) \cdot u_x(x, t_o)]$, $\rho(x)$, $c(x)$, and $u_t(x, t_o)$ it is found that the equation resulting from the limit is

$$\left\{ \frac{\partial}{\partial x} [K(x)\; u_x(x,t_o)] \right\} \Bigg|_{x = x_o} = \rho(x_o)\; c(x_o)\; u_t(x_o,t_o)\;.$$

But x_o and t_o are arbitrary. Consequently, for all $x \varepsilon [0,L]$ and $t \geq 0$,

$$\frac{\partial}{\partial x} [K(x)\; u_x(x,t)] = \rho(x)\; c(x)\; u_t(x,t)\;. \tag{56}$$

Thus, the result of the derivation is the well-known heat, or diffusion, equation with variable coefficients.

Electric Field Between Cylindrical Conductors

Consider a coaxial conductor consisting of two right cylinders C_o and C_1 whose traces in any plane perpendicular to the common axis of the cylinders are squares (Fig. 9). The open region A between the two cylinders is filled with a dielectric material. Furthermore, suppose

(1) that both cylinders, C_o and C_1, are equipotential surfaces with electric potentials V_o and V_1, respectively, where $V_1 > V_o$;

(2) that the cylinders are so long that end effects may be neglected;

(3) that the region A is charge free; and

(4) that the dielectric material in A is isotropic with constant permittivity ε.

For each point $(x,y,z) \varepsilon A$, let $V(x,y,z)$ measure the potential at (x,y,z). It is assumed that $V \varepsilon C^2$ in A. Let $\vec{D}(x,y,z) = (D_1(x,y,z),\; D_2(x,y,z),\; D_3(x,y,z))$ measure the electric displacement (or flux density) at $(x,y,z) \varepsilon A$, where the components of $\vec{D}$ are

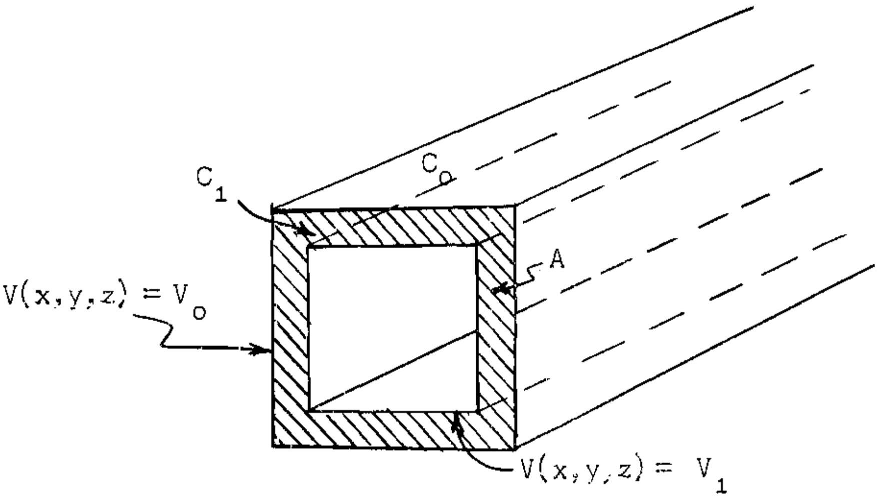

Figure 9. Coaxial Conductor

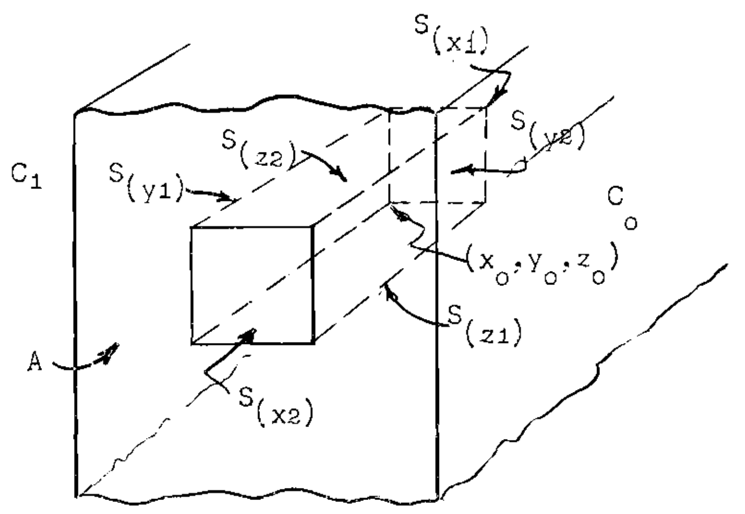

Figure 10. Rectangular Parallelepiped R in A

assumed to be of class C^2 in A. The functions $\vec{D}$ and V are related by the equation

$$\vec{D}(x,y,z) = -\varepsilon\,(\vec{\nabla}\,V)\Big|_{(x,y,z)} . \tag{57}$$

The discussion which follows is a derivation of Laplace's equation in three independent variables with V as the dependent variable. This equation along with suitable boundary conditions constitutes a mathematical model supposedly descriptive of the physical problem just outlined.

Let (x_o,y_o,z_o) be an arbitrarily chosen point in A. Consider a rectangular parallelepiped R of positive dimensions h, k, and l with one corner at (x_o,y_o,z_o) and contained entirely in A (Fig. 10). The flux through the face $S_{(x1)}$ is given by

$$\varphi_{(x1)} = \int_{S_{(x_1)}}\!\!\int D_1(x_o,y,z)\;dA\;, \tag{58}$$

and the flux through the face $S_{(x2)}$ is

$$\varphi_{(x2)} = \int_{S_{(x2)}}\!\!\int D_1(x_o + h,y,z)\;dA\;. \tag{59}$$

The difference represents the net flux $\varphi_{(x)}$ in the X-direction; thus,

$$\begin{aligned}\varphi_{(x)} &= \varphi_{(x2)} - \varphi_{(x1)} \\ &= \int_{S_{(x1)}}\!\!\int [D_1(x_o + h,y,z) - D_1(x_o,y,z)]\;dA,\end{aligned} \tag{60}$$

since $S_{(x1)}$ and $S_{(x2)}$ represent the same set of number pairs (y,z). Now there exists a function $\theta_1(y,z)$ such that if $(y,z) \in S_{(x1)}$, then $0 < \theta_1(y,z) < 1$, and

$$D_1(x_o + h, y, z) - D_1(x_o, y, z) = h\, D_{1x}(x_o + h\theta_1(y,z), y, z) .$$

Hence, Equation (60) may be rewritten as

$$\varphi_{(x)} = h \iint_{S_{(x1)}} D_{1x}(x_o + h\theta_1(y,z), y, z)\, dA . \qquad (61)$$

It was shown in Chapter IV that under the hypotheses stated, D_{1x} is a continuous function of all its arguments for $(y,z) \in S_{(x1)}$. Therefore, by Theorem 5

$$\iint_{S_{(x1)}} D_{1x}(x_o + h\theta_1(y,z), y, z)\, dA$$

$$= D_{1x}(x_o + h\theta_1(y_1, z_1), y_1, z_1)\, k\, l ,$$

where (y_1, z_1) is some point in $S_{(x1)}$. As a consequence of the last equation and (61) the equation

$$\varphi_{(x)} = h\, k\, l\, D_{1x}(x_o + h\theta_1(y_1, z_1), y_1, z_1) \qquad (62)$$

may be written, where $(y_1, z_1) \in S_{(x1)}$. Similarly, the net fluxes in the Y- and Z-directions are given by

$$\varphi_{(y)} = h\, k\, l\, D_{2y}(x_2, y_o + k\theta_2(x_2, z_2), z_2), \qquad (63)$$

for some $(x_2, z_2) \in S_{(y1)}$; and

$$\varphi_{(z)} = h\,k\,l\; D_{3z}(x_3, y_3, z_o + l\theta_3(x_3, y_3)), \tag{64}$$

for some $(x_3, y_3) \in S_{(z1)}$.

Since the parallelepiped R contains no charge (assumption 3),

$$\varphi_{(x)} + \varphi_{(y)} + \varphi_{(z)} = 0. \tag{65}$$

Thus, from Equations (62) - (65),

$$\begin{aligned} h\,k\,l\,[D_{1x}(x_o + h\theta_1(y_1, z_1), y_1, z_1) &+ D_{2y}(x_2, y_o + k\theta_2(x_2, y_2), z_2) \\ &+ D_{3z}(x_3, y_3, z_o + l\theta_3(x_3, y_3))] = 0; \end{aligned}$$

and, upon division by $h\,k\,l$,

$$\begin{aligned} D_{1x}(x_o + h\theta_1(y_1, z_1), y_1, z_1) &+ D_{2y}(x_2, y_o + k\theta_2(x_2, z_2), z_2) \\ &+ D_{3z}(x_3, y_3, z_o + l\theta_3(x_3, y_3)) = 0. \end{aligned} \tag{66}$$

If the limit as $(h, k, l) \rightarrow (0^+, 0^+, 0^+)$ is computed for Equation (66), the result is

$$D_{1x}(x_o, y_o, z_o) + D_{2y}(x_o, y_o, z_o) + D_{3z}(x_o, y_o, z_o) = 0,$$

or

$$(\vec{\nabla} \cdot \vec{D})\Big|_{(x_o, y_o, z_o)} = 0. \tag{67}$$

In combination Equations (57) and (67) imply that

$$\left[\vec{\nabla} \cdot (\varepsilon \vec{\nabla} V)\right]\Big|_{(x_0, y_0, z_0)} = 0,$$

which implies that

$$\left[\varepsilon \nabla^2 V\right]\Big|_{(x_0, y_0, z_0)} = 0,$$

or

$$\nabla^2 V (x_0, y_0, z_0) = 0 .$$

But (x_0, y_0, z_0) is an arbitrary point in A. Hence, for $(x, y, z) \varepsilon A$,

$$V_{xx}(x, y, z) + V_{yy}(x, y, z) + V_{zz}(x, y, z) = 0 .$$

This is Laplace's equation in three space dimensions and is the desired result.

CHAPTER VI

LOGICAL ERRORS AND SOME THOUGHTS ABOUT REMEDYING THEM

This chapter has two objectives: the first is to illustrate an error of logic typical of those which occur in various derivations of the mathematical models of physical systems; the second is to show how this error and others similar to it can be corrected. Two approaches to the problem of remedying such difficulties are presented. The basic structure of a derivation, though it may contain logical fallacies and careless omissions, is usually sound in principle and may be quite effective as far as intuitive appeal is concerned. Thus, one method by which a faulty derivation may be corrected consists of simply "patching up" the old derivation by replacing the defective portions. If this procedure is followed, then the original intuitive appeal is usually retained, but in a new logical setting. For this reason the "patching-up" procedure is often desirable, but it can lead to cumbersome mathematical operations. On the other hand it may be more desirable actually to construct a new mathematical derivation. In such cases physical intuition may play a less prominent part, but often a more elegant derivation will result.

The presentation in Chapter IV of the derivation of the dynamical equations for a perfect fluid in a three-dimensional region was motivated by a derivation of the same equations given in a well-known text of theoretical physics [3]. As in Chapter IV, the author of that text begins by appealing to a rectangular parallelepiped contained in the region under consideration. He then presents a rather informal mathematical discussion which terminates with the disclosure of the desired equations. In the course of the presentation, two deficiencies arise. The first, a quite commonly occurring one, is the failure to prescribe the analytic character of a function obviously assumed to have continuous first partial derivatives throughout the derivation. Otherwise, the derivation would be entirely useless. If the implicitly assumed continuity is granted, the author then appears to have made another error, only this time it is an error in mathematical logic. He seems to have adduced a mathematical assumption which is not valid in general -- namely, that a function of class C^1, which attains its average value on one face of the parallelepiped, say at (x',y',z'), assumes its average value on the opposite face at the point directly opposite (x',y',z'), that is, at $(x' + h, y',z')$. The following example shows the invalidity of his apparent assumption.

In the nomenclature of Chapter IV, let p be the pressure function defined by the equation

$$p(x,y,z,t) = x^2y + y^2 + z^2 ,$$

and let (x_1,y_1,z_1) be the origin $(0,0,0)$. Certainly $p \varepsilon C^1$ in $A = E_3$. The parallelepiped R (Fig. 6) has positive dimensions h, k, and l. Let t_1 be a fixed positive time. The average value $\overline{p}$ is computed by using Equation (29). Thus, since $x_1 = 0$,

$$\overline{p} = \frac{1}{k\,l}\iint_{S_1} p(0,y,z,t_1)\, dA =$$

$$= \frac{1}{k\,l} \int_0^l \int_0^k [y^2 + z^2]\, dy\, dz = \frac{k^2 + l^2}{3} .$$

This average value, $\bar{p}$, is assumed at points $(0,y,z)$ for which the equation

$$y^2 + z^2 = \frac{k^2 + l^2}{3}$$

and the inequalities $0 \leq y \leq k$ and $0 \leq z \leq l$ hold. Thus, $p(0,y,z,t_1) = \bar{p}$ on the face S_1 along the first quadrant arc of the circle with radius $\sqrt{k^2 + l^2/3}$ and center at the origin. Hereafter this locus is referred to as L_1.

The average value $\bar{\bar{p}}$ on the face S_2, opposite S_1, is computed in an analogous manner by using Equation (30). Thus, it is found that $\bar{\bar{p}} = \frac{h^2k}{2} + \frac{k^2}{3} + \frac{l^2}{3}$. The value $\bar{\bar{p}}$ is attained by p at points (h,y,z) for which

$$h^2y + y^2 + z^2 = \frac{h^2k}{2} + \frac{k^2}{3} + \frac{l^2}{3}$$

and the inequalities $0 \leq y \leq k$ and $0 \leq z \leq l$ hold. If the latter equation is written as

$$[y + \frac{h^2}{2}]^2 + z^2 = \frac{h^2k}{2} + \frac{k^2}{3} + \frac{l^2}{3} + \frac{h^4}{4} ,$$

then it is easily seen that $p(h,y,z,t_1) = \bar{\bar{p}}$ along that arc of the circle with center at $(h, -\frac{h}{2}, 0)$ and radius $\sqrt{\frac{h^2k}{2} + \frac{h^4}{4} + \frac{k^2}{3} + \frac{l^2}{3}}$ which lies in the face S_2. This locus is hereafter referred to as L_2.

It will now be shown that no point of L_2 lies opposite any point of L_1 for at least one choice of h, k, and l. The loci L_1 and L_2 may be considered as arcs of circles in the YZ-plane, and the corresponding problem is to show that L_1 and L_2 do not intersect. Since h, k, and l are arbitrary, it can be assumed that $h = k = l < \frac{3}{4}$. Suppose there were a point (y_o, z_o) which is a point of intersection of L_1 and L_2. Then the two equations

$$y_o^2 + z_o^2 = \frac{k^2 + l^2}{3} = \frac{2h^2}{3}$$

and

$$[y_o + \frac{h^2}{2}]^2 + z_o^2 = \frac{h^2 k}{2} + \frac{h^4}{4} + \frac{k^2}{3} + \frac{l^2}{3} = \frac{h^4}{4} + \frac{h^3}{2} + \frac{2h^2}{3}$$

must be valid. These two equations imply that

$$y_o = \frac{h}{2}.$$

But if (y_o, z_o) is a point of L_1, then

$$y_o < \frac{2h^2}{3}.$$

The latter equation and inequality imply that

$$\frac{h}{2} < \frac{2h^2}{3},$$

or

$$h > \frac{3}{4}.$$

But $h < \frac{3}{4}$. Hence L_1 and L_2 do not intersect, and the assumption cannot be made in general.

The derivation given in Chapter IV is then a "patching up" of the faulty derivation in the text mentioned. Recall that the derivation in Chapter IV is based on the consideration of a rectangular parallelepiped of positive dimensions h, k, and l. A large portion of that chapter is concerned with the derivation of Equation (37) for the net force exerted by the fluid on the parallelepiped in the X-direction. The derivation of this equation hinges primarily on finding an expression for the difference $\overline{p} - \overline{\overline{p}}$ between the average pressures on the opposite faces S_1 and S_2. This expression is then multiplied by the area kl of the faces S_1 and S_2 to obtain the net force. But the deduction of a reasonably manageable expression for $\overline{p} - \overline{\overline{p}}$ involves some rather awkward mathematical operations, even though this approach has intuitive appeal.

It is possible, however, if one is willing to sacrifice some intuition in order to gain a higher degree of mathematical elegance, to find an equation similar to (37) for the net force in the X-direction by much simpler means [1]. This new derivation is now presented in the framework of the same physical assumption and the same mathematical assumptions and nomenclature as those given at the beginning of Chapter IV. Thus, for fixed time t_o, let $f(y,z)$ be the function defined by the equation

$$f(y,z) = kl[p(x_o,y,z,t_o) - p(x_o+h,y,z,t_o)],$$

where $0 \leq y \leq k$, and $0 \leq z \leq l$. Now, if the pressures in the brackets are independent of y and z, then $f(y,z)$ is $\widetilde{F}_1$, the net force in the X-direction. Of course, in general this is not the case. In analogy to previous arguments of this nature, the logical model-building step, then, is to make the assumption

$$\underset{(y,z)\,\varepsilon\, S_1}{\text{Minimum}} \; [f(y,z)] \;\leq\; \tilde{F}_1 \;\leq\; \underset{(y,z)\,\varepsilon\, S_1}{\text{Maximum}} \; [f(y,z)] \; .$$

By Theorem 4 the continuity of f on S_1, which results from the continuity of p on S_1, then guarantees the existence of numbers ζ_1 and ζ_2 $(0 \leq \zeta_1 \leq 1)$, $0 \leq \zeta_2 \leq 1)$ such that

$$\tilde{F}_1 \;=\; f(y_0 + \zeta_1 k,\; z_0 + \zeta_2 l) \; ,$$

or

$$\tilde{F}_1 \;=\; kl[p(x_0, y_0 + \zeta_1 k, z_0 + \zeta_2 l, t_0) - p(x_0 + h, y_0 + \zeta_1 k, z_0 + \zeta_2 l, t_0)] \; .$$

A mean value theorem (Theorem 3) is applied with respect to the first argument in order to infer the existence of a number θ: $0 < \theta < 1$ such that

$$\tilde{F}_1 \;=\; -\,hkl\; p_x(x_0 + \theta h,\; y_0 + \zeta_1 k,\; z_0 + \zeta_2 l) \; ,$$

the new equation analogous to Equation (37). The remainder of the derivation remains as given in Chapter IV, but the derivation of the dynamical equations is greatly simplified in spite of the fact that some of the appeal to intuition has been lost.

Although the method presented in Chapter IV may not be the best for deriving the dynamical equations of a perfect fluid, it serves to contrast an effective but laborious method with a more elegant procedure;

and quite conceivably, the former approach may have more heuristic value in a completely unfamiliar problem.

APPENDIX

THEOREMS CITED IN THE TEXT

Theorem 1. Let f be a real-valued function which is continuous on a closed interval $[a,b]$ of the real line. Then there exist numbers x_m and x_M in $[a,b]$ such that $f(x) \geq f(x_m)$ and $f(x) \leq f(x_M)$ for all $x \varepsilon [a,b]$.

Theorem 2 (Intermediate Value Theorem). Let f be a real-valued and continuous function on a closed interval $[a,b]$ of the real line. Let c be any number such that $f(x_m) \leq c \leq f(x_M)$, where x_m and x_M are given in Theorem 1. Then there exists at least one number $\zeta \varepsilon [a,b]$ such that $f(\zeta) = c$.

Theorem 3 (Mean Value Theorem). Let f be real-valued and continuous on a closed interval $[a,b]$. Suppose further that $f'(x)$ exists for every $x \varepsilon (a,b)$. Then there exists at least one number $\zeta \varepsilon (a,b)$ such that

$$f(b) - f(a) = f'(\zeta) [b - a].$$

Theorem 4 (Intermediate Value Theorem). Let f be a real-valued function, defined and continuous on a connected set S in E_n. If f takes on two different values in S, say C and D, then for each number ζ between C and D there exists a point $\vec{x} \varepsilon S$ such that $f(\vec{x}) = \zeta$.

Theorem 5 (Mean Value Theorem for Multiple Integrals). Let f be a real-valued function defined and bounded on a bounded, Jordan-measurable set S in E_n. Let $m = \inf f(S)$ and $M = \sup f(s)$. Then there exists a real number λ in the interval $[m, M]$ such that

$$\int_S f(\vec{x})\, d\vec{x} = c(S) \cdot \lambda$$

where $c(S)$ is the Jordan measure of the set S.

Remark. Consider the case of a double integral of a continuous function over a bounded connected set S in E_2. Then λ is the "average value" of the function on S. By Theorem 4, there exists a point $(x_0, y_0) \in S$ such that $\lambda = f(x_0, y_0)$, and

$$\iint_S f(x, y)\, dA = f(x_0, y_0)\, A.$$

Thus f assumes its average value at a point $(x_0, y_0) \in S$, and this average value is given by

$$f(x_0, y_0) = \frac{1}{A} \iint_S f(x, y)\, dA, \qquad A \neq 0 .$$

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